International Journal of Solids and Structures 41 (2004) 2099–2110

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

A micromorphic electromagnetic theory

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Received 25 June 2002; received in revised form 20 August 2003

Abstract

This work is concerned with the determination of both macroscopic and microscopic deformations, motions, stresses, as well as electromagnetic fields developed in the material body due to external loads of thermal, mechanical, and electromagnetic origins. The balance laws of mass, microinertia, linear momentum, moment of momentum, energy, and entropy for microcontinuum are integrated with the Maxwell's equations. The constitutive theory is constructed. The finite element formulation of micromorphic electromagnetic physics is also presented. The physical meanings of various terms in the constitutive equations are discussed.

Keywords: Micromorphic; Electromagnetic; Balance laws; Constitutive relation; Finite element formulation

1. Introduction

Optical phonon branches exist in all crystals that have more than one atom per primitive unit cell. Under an electromagnetic field it is the optical modes that are excited. Optics is a phenomenon that necessitates the presence of an electromagnetic field.

While classical continuum theory is the long acoustic wave limit, lattice dynamics analysis has shown that micromorphic theory yields phonon dispersion relation similar to those from atomistic calculations and experimental measurements (Chen and Lee, 2003a). It provides up to 12 phonon dispersion relations, including 3 acoustic and 9 optical branches. The optical phonons in micromorphic theory describe the internal displacement patterns within the microstructure of material particles, consistent with the internal atomic displacements in the optical modes.

The physics of mechanical and electromagnetic coupling is hence related to the optical vibrations, and the continuum description of electrodynamics naturally leads to a micromorphic electromagnetic theory.

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doi:10.1016/j.ijsolstr.2003.11.031

2. Physical picture of micromorphic theories

Micromorphic Theory, developed by Eringen and Suhubi (1964) and Eringen (1999), constitutes extensions of the classical field theories concerned with the deformations, motions, and electromagnetic (E–M) interactions of material media, as continua, in microscopic time and space scales. In terms of a physical picture, a material body is envisioned as a collection of a large number of deformable particles; each particle possesses finite size and microstructure. The particle has the independent degrees of freedom for both stretches and rotations (micromorphic),and for rotations only (micropolar), in addition to the classical translational degrees of freedom of the center. It may be considered as a polyatomic molecule, a primitive unit cell of a crystalline solid, or a chopped fiber in a composite, et al. As shown in Fig. 1, a generic particle P is represented by its position vector $\boldsymbol{X}$ (mass center of P) and by a vector Ξ attached to P representing the microstructure of P in the reference state at time $t = 0$. The motions that carry $P(X, \Xi)$ to $P(\mathbf{x}, \boldsymbol{\xi}, t)$ in the spatial configuration (deformed state) at time t can be expressed as

$$x_k = x_k(\boldsymbol{X}, t), \tag{2.1}$$

$$\xi_k = \chi_{kK}(\boldsymbol{X}, t)\Xi_K. \tag{2.2}$$

It is seen that the macromotion, Eq. (2.1), accounts for the motion of the centroid of the particle while the micromotion, Eq. (2.2), specifies the changing of orientation and deformation for the inner structures of the particle. The inverse motions can be written as

$$X_K = X_K(\boldsymbol{x}, t), \tag{2.3}$$

$$\Xi_K = \bar{\chi}_{Kk}(\boldsymbol{x}, t)\xi_k, \tag{2.4}$$

where

$$\chi_{kK}\bar{\chi}_{Kl} = \delta_{kl}, \quad \bar{\chi}_{Kk}\chi_{kL} = \delta_{KL}. \tag{2.5}$$

3. The E–M balance laws

The balance laws of the micromorphic electromagnetic continuum consist of two parts: the thermo-mechanical part and the electromagnetic (E–M) parts. The E–M balance laws are the well-known Maxwell's equations written as

$$\nabla \cdot \boldsymbol{D} = q^{\mathrm{e}} \quad \text{or} \quad D_{k,k} = q^{\mathrm{e}}, \tag{3.1}$$

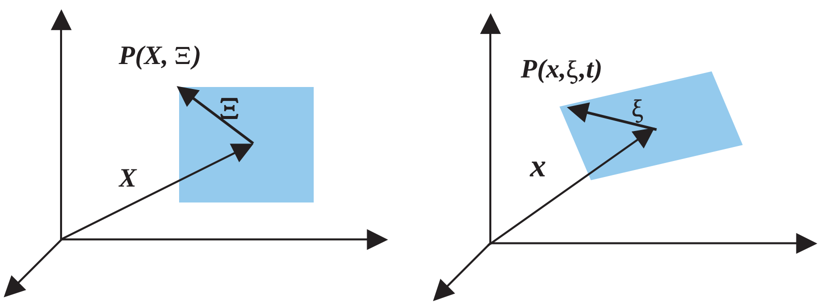

Fig. 1. The macro- and micro-motion of a material particle.

$$\nabla \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = 0 \quad \text{or} \quad e_{ijk}E_{k,j} + \frac{1}{c}\frac{\partial B_i}{\partial t} = 0, \tag{3.2}$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{or} \quad B_{k,k} = 0, \tag{3.3}$$

$$\nabla \times \boldsymbol{H} - \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} = \frac{1}{c}\boldsymbol{J} \quad \text{or} \quad e_{ijk}H_{k,j} - \frac{1}{c}\frac{\partial D_i}{\partial t} = \frac{1}{c}J_i, \tag{3.4}$$

where $\boldsymbol{D}$ is the dielectric displacement vector, $\boldsymbol{B}$ the magnetic flux vector, $\boldsymbol{E}$ the electric field vector, $\boldsymbol{H}$ the magnetic field vector, $\boldsymbol{J}$ the current vector, and q^e the free charge density. The divergence of Eq. (3.4) with the use of Eq. (3.1) leads to

$$\nabla \cdot \boldsymbol{J} + \frac{\partial q^e}{\partial t} = 0, \tag{3.5}$$

which is the law of conservation of charge. The divergence of Eq. (3.2) gives

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0$$

which is a duplicate of Eq. (3.3).

The polarization vector, $\boldsymbol{P}$, and the magnetization vector, $\boldsymbol{M}$, are defined as

$$\boldsymbol{P} \equiv \boldsymbol{D} - \boldsymbol{E}, \tag{3.6}$$

$$\boldsymbol{M} \equiv \boldsymbol{B} - \boldsymbol{H}. \tag{3.7}$$

The E–M vectors, $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{P}$, $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{M}$, $\boldsymbol{J}$, are all referred to a fixed laboratory frame R_C. The Galilean transformations of inertial frames form a group that consists of time-independent spatial rotations and pure Galilean transforms, i.e.,

$$x_i^* = R_{ij}x_j + V_i t + b_i, \tag{3.8}$$

where

$$R_{ik}R_{jk} = R_{ki}R_{kj} = \delta_{ij} \quad \text{and} \quad \det(\boldsymbol{R}) = 1. \tag{3.9}$$

The requirement of the form-invariance of the Maxwell's equations under the Galilean transformations leads to the following transformations (Eringen and Maugin, 1990)

$$q^{e^*} = q^e, \tag{3.10}$$

$$\boldsymbol{J}^* = \boldsymbol{J} - q^e\boldsymbol{v}, \tag{3.11}$$

$$\boldsymbol{P}^* = \boldsymbol{P}, \tag{3.12}$$

$$\boldsymbol{M}^* = \boldsymbol{M} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{P}, \tag{3.13}$$

$$\boldsymbol{E}^* = \boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}, \tag{3.14}$$

$$\boldsymbol{B}^* = \boldsymbol{B} - \frac{1}{c}\boldsymbol{v} \times \boldsymbol{E}, \tag{3.15}$$

$$\boldsymbol{D}^* = \boldsymbol{D} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}, \tag{3.16}$$

$$\boldsymbol{H}^* = \boldsymbol{H} - \frac{1}{c}\boldsymbol{v} \times \boldsymbol{D}, \tag{3.17}$$

where the quantities, q^{e^*}, $\boldsymbol{J}^*$, $\boldsymbol{P}^*$, $\boldsymbol{M}^*$, $\boldsymbol{E}^*$, $\boldsymbol{B}^*$, $\boldsymbol{D}^*$, $\boldsymbol{H}^*$, are referred to a co-moving frame R_G with material particles of the body having velocities, $\boldsymbol{v}$. A typical nonrelativistic feature of these transformations, Eqs. (3.10)–(3.17), is the asymmetry between Eq. (3.12) and Eq. (3.13), which says, according to Galilean relativity, a polarized moving body will appear to be magnetized, whereas a magnetized moving body will not appear to be polarized. Although it is this lack of symmetry that stimulated the study of relativistic electrodynamics in the early 20th century, few observable conclusions can be made due to the difficulty of obtaining sufficiently high velocities for material media. The fully symmetric relativistic laws replacing Eqs. (3.12) and (3.13) may be found in Jackson (1975).

4. The thermomechanical balance laws

The thermomechanical balance laws were originally obtained by Eringen and Suhubi (1964) by means of a "microscopic space-averaging" process. Later, Eringen (1999) re-derived the balance laws by starting with the following expression for the kinetic energy of a material particle

$$K = \frac{1}{2}\rho(v_i v_i + i_{kl}\upsilon_{ik}\upsilon_{il}), \tag{4.1}$$

and, after the energy balance law is obtained, by requiring it to be form-invariant under the generalized Galilean transformation to yield the balance laws of linear momentum and moment of momentum. Recently, Chen et al. (2002) and Chen and Lee (2003b,c) identified all the instantaneous mechanical variables, corresponding to those in micromorphic theory, in phase space; derived the corresponding field quantities in physical space through the statistical ensemble averaging process; invoked the time evolution law and the generalized Boltzmann transport equation for conserved properties to obtain the local balance laws of mass, microinertia, linear momentum, moment of momentum, and energy for microcontinuum field theory. In the case that the external field is the gravitational field, the balance laws obtained by Chen et al. (2002) and Chen and Lee (2003b,c) in a bottom-up approach agree perfectly with those obtained by Eringen and Suhubi (1964) and Eringen (1999) in a top–down approach.

The balance laws of micromorphic continuum with E–M interactions can be expressed as

$$\dot{\rho} + \rho\nabla\cdot\boldsymbol{v} = 0 \quad \text{or} \quad \dot{\rho} + \rho v_{i,i} = 0, \tag{4.2}$$

$$\frac{\mathrm{d}\boldsymbol{i}}{\mathrm{d}t} = \boldsymbol{i}\cdot\boldsymbol{\upsilon}' + \boldsymbol{\upsilon}\cdot\boldsymbol{i} \quad \text{or} \quad \frac{\mathrm{d}i_{kl}}{\mathrm{d}t} = i_{km}\upsilon_{lm} + i_{lm}\upsilon_{km}, \tag{4.3}$$

$$\nabla\cdot\boldsymbol{t} + \rho(\boldsymbol{f} - \dot{\boldsymbol{v}}) + \boldsymbol{F} = 0 \quad \text{or} \quad t_{ji,j} + \rho(f_i - \dot{v}_i) + F_i = 0, \tag{4.4}$$

$$\nabla\cdot\boldsymbol{\lambda} + \boldsymbol{t}' - \boldsymbol{s}' + \rho(\boldsymbol{l} - \boldsymbol{\sigma}) + \boldsymbol{L} = 0 \quad \text{or} \quad \lambda_{klm,k} + t_{ml} - s_{ml} + \rho(l_{lm} - \sigma_{lm}) + L_{lm} = 0, \tag{4.5}$$

$$\rho\dot{e} = \boldsymbol{\lambda}\,\vdots\,\nabla\boldsymbol{\upsilon} + \boldsymbol{t}:\nabla\boldsymbol{v} + (\boldsymbol{s} - \boldsymbol{t} - \boldsymbol{L})' : \boldsymbol{\upsilon} + \nabla\cdot\boldsymbol{q} + \rho h + w$$

or

$$\rho\dot{e} = \lambda_{klm}\upsilon_{lm,k} + t_{kl}v_{l,k} + (s_{kl} - t_{kl} - L_{kl})\upsilon_{lk} + q_{k,k} + \rho h + w, \tag{4.6}$$

where ρ, $\boldsymbol{v}$, $\boldsymbol{i}$, $\boldsymbol{\upsilon}$, $\boldsymbol{t}$, $\boldsymbol{s}=\boldsymbol{s}'$, $\boldsymbol{\lambda}$, e, $\boldsymbol{q}$ are the mass density, velocity, microinertia, microgyration, Cauchy stress, microstress average, moment stress, internal energy, and heat input, respectively; $\boldsymbol{f}$,$\boldsymbol{l}$, h represent body force, body moment and heat source of mechanical origin, respectively. The spin inertia $\boldsymbol{\sigma}$ is defined as

$$\sigma_{kl} = i_{ml}(\dot{\upsilon}_{km} + \upsilon_{kn}\upsilon_{nm}) \tag{4.7}$$

and the body force, body moment, and energy source of E–M origin are given as (Eringen and Maugin, 1990; Eringen, 1999; De Groot and Suttorp, 1972):

$$\boldsymbol{F} = q^{e}\boldsymbol{E}^{*} + (\boldsymbol{P}\cdot\nabla)\boldsymbol{E}^{*} + (\nabla\boldsymbol{B})\cdot\boldsymbol{M}^{*} + \frac{1}{c}\{\boldsymbol{J}^{*} + \dot{\boldsymbol{P}} - (\boldsymbol{P}\cdot\nabla)\boldsymbol{v} + \boldsymbol{P}(\nabla\cdot\boldsymbol{v})\}\times\boldsymbol{B}, \tag{4.8}$$

$$\boldsymbol{L} = \boldsymbol{P}\otimes\boldsymbol{E}^{*} - \boldsymbol{B}\otimes\boldsymbol{M}^{*}, \tag{4.9}$$

$$W = \boldsymbol{E}^{*}\cdot(\dot{\boldsymbol{P}} + \boldsymbol{P}\nabla\cdot\boldsymbol{v}) - \boldsymbol{M}^{*}\cdot\dot{\boldsymbol{B}} + \boldsymbol{J}^{*}\cdot\boldsymbol{E}^{*}. \tag{4.10}$$

The second law of thermodynamics, also referred to as the Clausius–Duhem inequality, is written as

$$\rho\dot{\eta} - \nabla\cdot(\boldsymbol{q}/\theta) - \rho h/\theta \geqslant 0, \tag{4.11}$$

where η is the entropy density and θ the absolute temperature. Now the generalized Helmholtz's free energy ψ is introduced as

$$\psi \equiv e - \theta\eta - \boldsymbol{E}^{*}\cdot\boldsymbol{P}/\rho. \tag{4.12}$$

Then the Clausius–Duhem (C–D) inequality can be expressed as

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + \lambda_{ijk}\upsilon_{jk,i} + t_{ij}v_{j,i} + (s_{ij} - t_{ij} - P_iE_j^{*} + B_iM_j^{*})\upsilon_{ji} + \frac{1}{\theta}q_i\theta_{,i} - P_i\dot{E}_i^{*} - M_i^{*}\dot{B}_i + J_i^{*}E_i^{*} \geqslant 0. \tag{4.13}$$

5. Constitutive relations

The fundamental laws of micromorphic electromagnetic continuum consist of a system of 27 partial differential equations, Eqs. (3.1)–(3.4), (4.2)–(4.6), and one inequality, Eq. (4.13). There are 83 unknowns: ρ, i_{kl}, v_k, υ_{kl}, θ, η, e, t_{kl}, $s_{kl}=s_{lk}$, λ_{klm}, q_k, q^{e}, E_k, P_k, B_k, M_k, J_k, considering that the body force, body moment, and heat source are given. Therefore 56 constitutive relations are needed to determine the dynamics of the thermomechanical-electromagnetic system.

The generalized Lagrangian strain tensors of micromorphic theory are defined as

$$\alpha_{KL} \equiv x_{k,K}\bar{\chi}_{Lk} - \delta_{KL}, \tag{5.1}$$

$$\beta_{KL} \equiv \chi_{kK}\chi_{kL} - \delta_{KL} = \beta_{LK}, \tag{5.2}$$

$$\gamma_{KLM} \equiv \bar{\chi}_{Kk}\chi_{kL,M}, \tag{5.3}$$

and the strain rates can be obtained as

$$\dot{\alpha}_{KL} \equiv (v_{l,k} - \upsilon_{lk})x_{k,K}\bar{\chi}_{Ll}, \tag{5.4}$$

$$\dot{\beta}_{KL} = (\upsilon_{kl} + \upsilon_{lk})\chi_{kK}\chi_{lL} = \dot{\beta}_{LK}, \tag{5.5}$$

$$\dot{\gamma}_{KLM} = \upsilon_{kl,m}\bar{\chi}_{Kk}\chi_{lL}x_{m,M}. \tag{5.6}$$

It can be easily proved that the Lagrangian strains and their material time rates of any order are objective, and hence they are suitable for being employed as independent constitutive variables in the development of a constitutive theory. In the same spirit, define the Lagrangian forms of the electric field vector and the magnetic flux vector as

$$E_K^* \equiv E_k^* x_{k,K}, \tag{5.7}$$

$$B_K = B_k x_{k,K}, \tag{5.8}$$

and their material time rates are obtained

$$\dot{E}_K^* = (\dot{E}_k^* + E_l^* v_{l,k}) x_{k,K}, \tag{5.9}$$

$$\dot{B}_K = (\dot{B}_k + B_l v_{l,k}) x_{k,K}, \tag{5.10}$$

The generalized 2nd order Piola–Kirchhoff stress tensors of micromorphic theory are defined as

$$T_{KL} \equiv j t_{kl} X_{k,k} \chi_{lL}, \tag{5.11}$$

$$S_{KL} \equiv j s_{kl} \bar{\chi}_{Kk} \bar{\chi}_{Ll}/2, \tag{5.12}$$

$$\Gamma_{KLM} \equiv j \lambda_{mkl} X_{M,m} \chi_{kK} \bar{\chi}_{Ll}, \tag{5.13}$$

where $j \equiv \det(x_{k,K})$ is the Jacobian of the deformation gradient. It is straightforward to show

$$T_{KL}\dot{\alpha}_{KL} + S_{KL}\dot{\beta}_{KL} + \Gamma_{KLM}\dot{\gamma}_{KLM} = j\{t_{kl}(v_{l,k} - v_{lk}) + s_{kl}v_{(kl)} + \lambda_{klm}v_{lm,k}\}, \tag{5.14}$$

which means $\{\boldsymbol{T}, \boldsymbol{S}, \boldsymbol{\Gamma}\}$ are the thermodynamic conjugates of $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$. Similarly, the Lagrangian forms of the heat input, polarization, magnetization, and current are defined as

$$Q_K \equiv j q_k X_{K,k}, \tag{5.15}$$

$$P_K \equiv j P_k X_{K,k}, \tag{5.16}$$

$$M_K^* \equiv j M_k^* X_{K,k}, \tag{5.17}$$

$$J_K^* \equiv j J_k^* X_{K,k}. \tag{5.18}$$

Now, the Clausius–Duhem inequality (4.13) can be rewritten as

$$-\rho^0(\dot{\psi} + \eta\dot{\theta}) + \Gamma_{KLM}\dot{\gamma}_{KLM} + T_{KL}^{\mathrm{m}}\dot{\alpha}_{KL} + S_{KL}^{\mathrm{m}}\dot{\beta}_{KL} + \frac{1}{\theta}Q_K\theta_{,K} - P_K\dot{E}_K^* - M_K^*\dot{B}_K + J_K^*E_K^* \geqslant 0, \tag{5.19}$$

where

$$t_{kl}^{\mathrm{m}} \equiv t_{kl} + P_k E_l^* + M_k^* B_l, \tag{5.20}$$

$$s_{kl}^{\mathrm{m}} \equiv s_{kl} + M_k^* B_l + M_l^* B_k = s_{lk}^{\mathrm{m}}, \tag{5.21}$$

$$T_{KL}^{\mathrm{m}} \equiv j t_{kl}^{\mathrm{m}} X_{K,k} \chi_{lL}, \tag{5.22}$$

$$S_{KL}^{\mathrm{m}} \equiv j s_{kl}^{\mathrm{m}} \bar{\chi}_{Kk} \bar{\chi}_{Ll}/2, \tag{5.23}$$

where the superscript ‘m’ refers to the mechanical parts, i.e., if there is no E–M interaction, then $S_{KL} = S_{KL}^{\mathrm{m}}$, $t_{kl} = t_{kl}^{\mathrm{m}}$. Note that the mechanical part of the microstress average $\boldsymbol{S}^{\mathrm{m}}$ is also symmetric.

In this work, the independent and dependent constitutive variables are set to be

$$\boldsymbol{Y} = \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \theta, \nabla\theta, \boldsymbol{E}^*, \boldsymbol{B}, \boldsymbol{X}\}, \tag{5.24}$$

$$\boldsymbol{Z} = \{\boldsymbol{T}^{\mathrm{m}}, \boldsymbol{S}^{\mathrm{m}}, \boldsymbol{\Gamma}, \psi, \eta, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{M}^*, \boldsymbol{J}^*\}, \tag{5.25}$$

and, following the axiom of equipresence, at the outset the constitutive relations are written as

$$\boldsymbol{Z} = \boldsymbol{Z}(\boldsymbol{Y}). \tag{5.26}$$

It is noted that there are 56 dependent constitutive variables in $\boldsymbol{Z}$. Both $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are presented in Lagrangian forms, hence, the axiom of objectivity is automatically satisfied. Substituting Eq. (5.26) into the C–D inequality (5.19), it follows

$$\begin{aligned} &-\rho^\circ\left(\frac{\partial\psi}{\partial\theta}+\eta\right)\dot{\theta} - \rho^\circ\frac{\partial\psi}{\partial\theta_{,K}}\dot{\theta}_{,K} + \left(T^{\mathrm{m}}_{KL} - \rho^\circ\frac{\partial\psi}{\partial\alpha_{KL}}\right)\dot{\alpha}_{KL} + \left(S^{\mathrm{m}}_{KL} - \rho^\circ\frac{\partial\psi}{\partial\beta_{KL}}\right)\dot{\beta}_{KL} + \left(\Gamma_{KLM} - \rho^\circ\frac{\partial\psi}{\partial\gamma_{KLM}}\right)\dot{\gamma}_{KLM} \\ &\quad - \left(P_K + \rho^\circ\frac{\partial\psi}{\partial E^*_K}\right)\dot{E}^*_K - \left(M^*_K + \rho^\circ\frac{\partial\psi}{\partial B_K}\right)\dot{B}_K + \frac{1}{\theta}Q_K\theta_{,K} + J^*_K E^*_K \geqslant 0. \end{aligned} \tag{5.27}$$

Since the inequality (5.27) is linear in $\dot{\theta}$, $\nabla\dot{\theta}$, $\dot{\boldsymbol{\alpha}}$, $\dot{\boldsymbol{\beta}}$, $\dot{\boldsymbol{\gamma}}$, $\dot{\boldsymbol{E}}^*$, $\dot{\boldsymbol{B}}$, it holds if and only if

$$\psi = \psi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \theta, \boldsymbol{E}^*, \boldsymbol{B}, \boldsymbol{X}), \tag{5.28}$$

$$\eta = -\frac{\partial\psi}{\partial\theta}, \tag{5.29}$$

$$\boldsymbol{T}^{\mathrm{m}} = \rho^\circ\frac{\partial\psi}{\partial\boldsymbol{\alpha}}, \tag{5.30}$$

$$\boldsymbol{S}^{\mathrm{m}} = \rho^\circ\frac{\partial\psi}{\partial\boldsymbol{\beta}}, \tag{5.31}$$

$$\boldsymbol{\Gamma} = \rho^\circ\frac{\partial\psi}{\partial\boldsymbol{\gamma}}, \tag{5.32}$$

$$\boldsymbol{P} = -\rho^\circ\frac{\partial\psi}{\partial\boldsymbol{E}^*}, \tag{5.33}$$

$$\boldsymbol{M}^* = -\rho^\circ\frac{\partial\psi}{\partial\boldsymbol{B}}, \tag{5.34}$$

$$\boldsymbol{Q}\cdot\nabla\theta + \theta\boldsymbol{J}^*\cdot\boldsymbol{E}^* \geqslant 0, \tag{5.35}$$

these constitutive relations, Eqs. (5.28)–(5.35), are further subjected to the axioms of material invariance and time reversal. It may be stated as: the constitutive response functionals must be form-invariant with respect to a group of transformations of the material frame of reference $\{\boldsymbol{X} \to \boldsymbol{X}^*\}$ and microscopic time reversal $\{t \to -t\}$ representing the material symmetry conditions and these transformations must leave the density and charge at $(\boldsymbol{X}, t)$ unchanged (Eringen and Maugin, 1990). The magnetic symmetry properties of solids cannot be discussed rationally by means of three-dimensional point groups only since magnetism is the result of the spin magnetic moment of electrons, which changes sign upon the time reversal. In other words, diamagnetic and paramagnetic crystals do not exhibit any orderly distribution of their spin magnetic moments, and are therefore 'time symmetric'. The crystallographic point group is enough for the discussion of their material symmetries; on the other hand, for ferromagnetic, ferrimagnetic and antiferromagnetic materials, which are characterized by an orderly distribution of spin magnetic moment, an additional symmetry operator is needed to take care of the time reversal. For a complete account of this subject, interested readers are referred to Shubnikov and Belov (1964) and Kiral and Eringen (1990).

6. Finite element formulation

The energy equation (4.6) can now be written as

$$\rho^\circ \theta \dot{\eta} - Q_{K,K} - \rho^\circ h - J_K^* E_K^* = 0, \tag{6.1}$$

Multiply Eq. (6.1) by the variational temperature $\delta\theta$ and then integrate over the undeformed volume, to yield

$$\int_V \rho^\circ \theta \dot{\eta} \delta\theta \, \mathrm{d}V + \int_V Q_K \delta\theta_{,K} \, \mathrm{d}V = \int_V \Phi \delta\theta \, \mathrm{d}V + \int_{S_q} Q^* \delta\theta \, \mathrm{d}S, \tag{6.2}$$

where

$$\Phi \equiv \rho^\circ h + J_K^* E_K^*, \tag{6.3}$$

and

$$Q^* \equiv Q_K N_K, \tag{6.4}$$

is the heat input specified at S_q, part of the surface S that enclosing the volume V and N_K is the outward normal to S. It is noted that $S_q \cup S_\theta = S$, where S_θ is part of the surface on which the temperature is specified.

The balance laws of linear momentum and moment of momentum, Eqs. (4.4) and (4.5), can be expressed in the Lagrangian forms as

$$(T_{KL}^{\mathrm{m}} \bar{\chi}_{Li})_{,K} + \rho^\circ (\tilde{f}_i - \dot{v}_i) = 0 \tag{6.5}$$

$$(\Gamma_{LMK} \bar{\chi}_{Li} \chi_{jM})_{,K} + j(t_{ji}^{\mathrm{m}} - s_{ji}^{\mathrm{m}}) + \rho^\circ (l_{ij} - \sigma_{ij}) = 0 \tag{6.6}$$

where

$$\tilde{f}_i = f_i + \{F_i - (P_j E_i^* + M_j^* B_i)_{,j}\}/\rho. \tag{6.7}$$

Multiply Eq. (6.5) and Eq. (6.6) by the variational velocity vector δv_i and the variational microgyration tensor $\delta \upsilon_{ij}$, respectively, and then integrate the sum over the undeformed volume, to get

$$\begin{aligned} &\int_V \{\Gamma_{KLM} \delta\dot{\gamma}_{KLM} + T_{KL}^{\mathrm{m}} \delta\dot{\alpha}_{KL} + S_{KL}^{\mathrm{m}} \delta\dot{\beta}_{KL}\} \, \mathrm{d}V + \int_V \rho^\circ \{\dot{v}_i \delta v_i + \sigma_{ij} \delta\upsilon_{ij}\} \, \mathrm{d}V \\ &\quad = \int_V \rho^\circ \{\tilde{f}_i \delta v_i + \tilde{l}_{ij} \delta\upsilon_{ij}\} \, \mathrm{d}V + \int_{S_t} T_i^* \delta v_i \, \mathrm{d}S + \int_{S_\Gamma} \Gamma_{ij}^* \delta\upsilon_{ij} \, \mathrm{d}S, \end{aligned} \tag{6.8}$$

where

$$T_i^* \equiv T_{KL}^{\mathrm{m}} \bar{\chi}_{Li} N_K, \tag{6.9}$$

$$\Gamma_{ij}^* \equiv \Gamma_{LMK} \bar{\chi}_{Li} \chi_{jM} N_K, \tag{6.10}$$

are the surface load and surface moment specified at S_t and S_Γ, respectively. It is noted that

$$S = S_t \cup S_v = S_\Gamma \cup S_\upsilon, \tag{6.11}$$

where the velocity and the microgyration are specified on S_v and S_υ, respectively. In this finite element formulation no restrictive assumption has been made to the magnitude of any independent constitutive variables. The results are valid for coupled thermomechanical-electromagnetic phenomena. It is seen, from Eqs. (6.1) and (6.8), that to proceed further one needs the explicit constitutive expressions for the entropy η, the heat input vector $\boldsymbol{Q}$, and the generalized 2nd order Piola–Kirchhoff stress tensor $\boldsymbol{T}^{\mathrm{m}}$, $\boldsymbol{S}^{\mathrm{m}}$, and $\boldsymbol{\Gamma}$.

7. Linear constitutive equations

To derive the linear constitutive equations for micromorphic electromagnetic continuum, first, let the Helmholtz's free energy density, Eq. (5.28), be expanded as a polynomial up to second order in terms of its arguments

$$\begin{aligned}\rho^0\psi = {} & \rho^0\psi^0 - \rho^0\eta^0 T + T^0_{KL}\alpha_{KL} + S^0_{KL}\beta_{KL} + \Gamma^0_{KLM}\gamma_{KLM} - P^0_K E^*_K - M^0_K B_K - \frac{1}{2}\rho^0\gamma T^2/T^0 - a^1_{KL}T\alpha_{KL} - a^2_{KL}T\beta_{KL} \\ & - a^3_{KLM}T\gamma_{KLM} - a^4_K T E^*_K - a^5_K T B_K + \frac{1}{2}A^1_{KLMN}\alpha_{KL}\alpha_{MN} + A^4_{KLMN}\alpha_{KL}\beta_{MN} + A^5_{KLMNP}\alpha_{KL}\gamma_{MNP} \\ & - B^1_{KLM}\alpha_{KL}E^*_M - C^1_{KLM}\alpha_{KL}B_M + \frac{1}{2}A^2_{KLMN}\beta_{KL}\beta_{MN} + A^6_{KLMNP}\beta_{KL}\gamma_{MNP} - B^2_{KLM}\beta_{KL}E^*_M \\ & - C^2_{KLM}\beta_{KL}B_M + \frac{1}{2}A^3_{KLMNPQ}\gamma_{KLM}\gamma_{NPQ} - B^3_{KLMN}\gamma_{KLM}E^*_N - C^3_{KLMN}\gamma_{KLM}B_N - \frac{1}{2}D^1_{KL}E^*_K E^*_L \\ & - \frac{1}{2}D^2_{KL}B_K B_L - D^3_{KL}E^*_K B_L, \end{aligned} \tag{7.1}$$

where T^0 is the reference temperature,

$$\theta = T^0 + T, \quad T^0 > 0, \ |T| \ll T^0, \tag{7.2}$$

$$A^1_{KLMN} = A^1_{MNKL}, \tag{7.3}$$

$$A^2_{KLMN} = A^2_{MNKL} = A^2_{LKMN} = A^2_{KLNM}, \tag{7.4}$$

$$A^3_{KLMNPQ} = A^3_{NPQKLM}, \tag{7.5}$$

$$A^4_{KLMN} = A^4_{KLNM}, \tag{7.6}$$

$$A^6_{KLMNP} = A^6_{LKMNP}, \tag{7.7}$$

$$S^0_{KL} = S^0_{LK}, \tag{7.8}$$

$$a^2_{KL} = a^2_{LK}, \tag{7.9}$$

$$B^2_{KLM} = B^2_{LKM}, \tag{7.10}$$

$$C^2_{KLM} = C^2_{LKM}, \tag{7.11}$$

$$D^1_{KL} = D^1_{LK}, \tag{7.12}$$

$$D^2_{KL} = D^2_{LK}. \tag{7.13}$$

Then Eqs. (5.29)–(5.34) leads to

$$\eta = \eta^0 + \gamma T/T^0 + \{a^1_{KL}\alpha_{KL} + a^2_{KL}\beta_{KL} + a^3_{KLM}\gamma_{KLM} + a^4_K E^*_K + a^5_K B_K\}/\rho^0, \tag{7.14}$$

$$T^{\mathrm{m}}_{KL} = T^0_{KL} - a^1_{KL}T + A^1_{KLMN}\alpha_{MN} + A^4_{KLMN}\beta_{MN} + A^5_{KLMNP}\gamma_{MNP} - B^1_{KLM}E^*_M - C^1_{KLM}B_M, \tag{7.15}$$

$$S^{\mathrm{m}}_{KL} = S^0_{KL} - a^2_{KL}T + A^2_{KLMN}\beta_{MN} + A^4_{MNKL}\alpha_{MN} + A^6_{KLMNP}\gamma_{MNP} - B^2_{KLM}E^*_M - C^2_{KLM}B_M, \tag{7.16}$$

$$\Gamma_{KLM} = \Gamma^0_{KLM} - a^3_{KLM}T + A^3_{KLMNPQ}\gamma_{NPQ} + A^5_{NPKLM}\alpha_{NP} + A^6_{NPKLM}\beta_{NP} - B^3_{KLMN}E^*_N - C^3_{KLMN}B_N, \tag{7.17}$$

$$P_K = P^0_K + a^4_K T + B^1_{LMK}\alpha_{LM} + B^2_{LMK}\beta_{LM} + B^3_{LMNK}\gamma_{LMN} + D^1_{KL}E^*_L + D^3_{KL}B_L, \tag{7.18}$$

$$M^*_K = M^0_K + a^5_K T + C^1_{LMK}\alpha_{LM} + C^2_{LMK}\beta_{LM} + C^3_{LMNK}\gamma_{LMN} + D^2_{KL}B_L + D^3_{LK}E^*_L, \tag{7.19}$$

where η^0, $\{\boldsymbol{T^0}, \boldsymbol{S^0}, \boldsymbol{\Gamma^0}\}$, $\boldsymbol{P^0}$, $\boldsymbol{M^0}$ are the initial entropy, stresses, polarization, and magnetization, respectively; $\boldsymbol{a}^1$, $\boldsymbol{a}^2$, $\boldsymbol{a}^3$ are the thermal stresses moduli; $\boldsymbol{a}^4$, $\boldsymbol{a}^5$ are the pyroelectric and pyromagnetic moduli; γ is the heat capacity; $\boldsymbol{A^i}$ $(i = 1, 2, \ldots, 6)$ are the generalized elastic moduli; $\boldsymbol{B}^1$, $\boldsymbol{B}^2$, $\boldsymbol{B}^3$ are the generalized piezoelectric moduli; $\boldsymbol{C}^1$, $\boldsymbol{C}^2$, $\boldsymbol{C}^3$ are the generalized piezomagnetic moduli; $\boldsymbol{D}^1$ is the dielectric susceptibility; $\boldsymbol{D}^2$ is the magnetic susceptibility; $\boldsymbol{D}^3$ is the magnetic polarizability.

Now, in view of the Clausius–Duhem inequality (5.35), the linear constitutive equations for the heat input and the current can be obtained as

$$Q_K = H^1_{KL}\theta_{,L} + H^3_{KL}E^*_L, \tag{7.20}$$

$$\theta J^*_K = H^2_{KL}E^*_L + H^4_{KL}\theta_{,L}, \tag{7.21}$$

where $\boldsymbol{H}^1$ is the heat conductivity, $\boldsymbol{H}^2$ is the electric conductivity, $\boldsymbol{H}^3$ indicates the Peltier effect, $\boldsymbol{H}^4$ indicates the Seebeck effect. If the Onsager postulate is followed, then there exists a dissipation function $\Phi(\frac{\nabla\theta}{\theta}, \boldsymbol{E}^*)$ which is nonnegative with an absolute minimum at $\nabla\theta = \boldsymbol{E}^* = 0$ and yields

$$Q_K = \frac{\partial\Phi}{\partial(\nabla\theta/\theta)}, \tag{7.22}$$

$$J^*_K = \frac{\partial\Phi}{\partial E^*_K}, \tag{7.23}$$

This implies

$$H^1_{KL} = H^1_{LK}, \quad H^2_{KL} = H^2_{LK}, \tag{7.24}$$

$$H^3_{KL} = H^4_{LK} \equiv G_{KL}, \tag{7.25}$$

and

$$\boldsymbol{H} \equiv \begin{vmatrix} \boldsymbol{H}^1 & \boldsymbol{G} \\ \boldsymbol{G}' & \boldsymbol{H}^2 \end{vmatrix} \tag{7.26}$$

is positive definite. All the above-mentioned material moduli may be functions of the Lagrangian coordinate $\boldsymbol{X}$ and the reference temperature T^0.

On the other hand, from Eqs. (5.24)–(5.26), it is seen that in general $\boldsymbol{Q}$ and $\boldsymbol{J}^*$ are functions of the three generalized Lagrangian strains, temperature, temperature gradient, electric field, magnetic flux, and the Lagrangian coordinate. For isotropic material in a simpler case, i.e., neglecting the effect of the third order strain tensor $\boldsymbol{\gamma}$, the current and the heat input can be rewritten as

$$\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \nabla\boldsymbol{\theta}, \boldsymbol{E}^*, \boldsymbol{B}, \boldsymbol{S}, \boldsymbol{X}), \tag{7.27}$$

$$\boldsymbol{J}^* = \boldsymbol{J}^*(\boldsymbol{\varepsilon}, \boldsymbol{\beta}, \nabla\boldsymbol{\theta}, \boldsymbol{E}^*, \boldsymbol{B}, \boldsymbol{S}, \boldsymbol{X}), \tag{7.28}$$

where

$$\varepsilon_{KL} \equiv \frac{1}{2}(\alpha_{KL} + \alpha_{LK} + \beta_{KL}) = \varepsilon_{LK}, \tag{7.29}$$

$$S_K \equiv \frac{1}{2} e_{KLM} \alpha_{[LM]}. \tag{7.30}$$

It should be remarked that (1) ε_{KL} will be reduced to $u_{(K,L)}$—the classical macro-strain tensor in the case of small deformation, and (2) $\boldsymbol{B}$ and $\boldsymbol{S}$ are axial vectors and transformed as

$$\bar{\boldsymbol{B}} = \boldsymbol{R}\boldsymbol{B}\det(\boldsymbol{R}), \quad \boldsymbol{S} = \boldsymbol{R}\boldsymbol{S}\det(\boldsymbol{R}), \tag{7.31}$$

while the absolute vectors $\boldsymbol{E}^*$ and $\nabla\theta$ are transformed as

$$\bar{\nabla}|\theta \equiv \frac{\partial\theta}{\partial\bar{X}} = \boldsymbol{R}\nabla\theta, \quad \bar{\boldsymbol{E}}^* = \boldsymbol{R}\boldsymbol{E}^*, \tag{7.32}$$

where

$$\bar{\boldsymbol{X}} = \boldsymbol{R}\boldsymbol{X}. \tag{7.33}$$

Now, according to Wang's representation theorem for isotropic functions (Wang, 1970, 1971), it follows

$$\begin{aligned}
\boldsymbol{Q} = {} & c_1\boldsymbol{E}^* + c_2\nabla\theta + c_3\boldsymbol{\varepsilon}\boldsymbol{E}^* + c_4\boldsymbol{\varepsilon}\nabla\theta + c_5\boldsymbol{\varepsilon}^2\boldsymbol{E}^* + c_6\boldsymbol{\varepsilon}^2\nabla\theta + c_7\boldsymbol{\beta}\boldsymbol{E}^* + c_8\boldsymbol{\beta}\nabla\theta + c_9\boldsymbol{\beta}^2\boldsymbol{E}^* + c_{10}\boldsymbol{\beta}^2\nabla\theta \\
& + c_{11}\boldsymbol{B}\times\boldsymbol{E}^* + \boldsymbol{c}_{12}\boldsymbol{B}\times\nabla\theta + c_{13}\boldsymbol{S}\times\boldsymbol{E}^* + c_{14}\boldsymbol{S}\times\nabla\theta + c_{15}(\boldsymbol{\varepsilon}\boldsymbol{\beta} - \boldsymbol{\beta}\boldsymbol{\varepsilon})\boldsymbol{E}^* + c_{16}(\boldsymbol{\varepsilon}\boldsymbol{\beta} - \boldsymbol{\beta}\boldsymbol{\varepsilon})\nabla\theta \\
& + c_{17}[(\boldsymbol{B}\cdot\boldsymbol{B})\boldsymbol{E}^* - (\boldsymbol{B}\cdot\boldsymbol{E}^*)\boldsymbol{B}] + c_{18}[(\boldsymbol{B}\cdot\boldsymbol{B})\nabla\theta - (\boldsymbol{B}\cdot\nabla\theta)\boldsymbol{B}] + c_{19}[(\boldsymbol{S}\cdot\boldsymbol{S})\boldsymbol{E}^* - (\boldsymbol{S}\cdot\boldsymbol{E}^*)\boldsymbol{S}] \\
& + c_{20}[(\boldsymbol{S}\cdot\boldsymbol{S})\nabla\theta - (\boldsymbol{S}\cdot\nabla\theta)\boldsymbol{S}] + c_{21}[(\boldsymbol{B}\cdot\boldsymbol{E}^*)\boldsymbol{S} - (\boldsymbol{S}\cdot\boldsymbol{E}^*)\boldsymbol{B}] + c_{22}[(\boldsymbol{B}\cdot\nabla\boldsymbol{\theta})\boldsymbol{S} - (\boldsymbol{S}\cdot\nabla\theta)\boldsymbol{B}] \\
& + c_{23}[\boldsymbol{\varepsilon}(\boldsymbol{E}^*\times\boldsymbol{B}) - (\boldsymbol{\varepsilon}\boldsymbol{E}^*)\times\boldsymbol{B}] + c_{24}[\boldsymbol{\varepsilon}(\nabla\theta\times\boldsymbol{B}) - (\boldsymbol{\varepsilon}\nabla\boldsymbol{\theta})\times\boldsymbol{B}] + c_{25}[\boldsymbol{\varepsilon}(\boldsymbol{E}^*\times\boldsymbol{S}) - (\boldsymbol{\varepsilon}\boldsymbol{E}^*)\times\boldsymbol{S}] \\
& + c_{26}[\boldsymbol{\varepsilon}(\nabla\theta\times\boldsymbol{S}) - (\boldsymbol{\varepsilon}\nabla\boldsymbol{\theta})\times\boldsymbol{S}] + c_{27}[\boldsymbol{\beta}(\boldsymbol{E}^*\times\boldsymbol{B}) - (\boldsymbol{\beta}\boldsymbol{E}^*)\times\boldsymbol{B}] + c_{28}[\boldsymbol{\beta}(\nabla\theta\times\boldsymbol{B}) - (\boldsymbol{\beta}\nabla\theta)\times\boldsymbol{B}] \\
& + c_{29}[\boldsymbol{\beta}(\boldsymbol{E}^*\times\boldsymbol{S}) - (\boldsymbol{\beta}\boldsymbol{E}^*)\times\boldsymbol{S}] + c_{30}[\boldsymbol{\beta}(\nabla\theta\times\boldsymbol{S}) - (\boldsymbol{\beta}\nabla\theta)\times\boldsymbol{S}],
\end{aligned} \tag{7.34}$$

$$\begin{aligned}
\boldsymbol{J}^* = {} & d_1\boldsymbol{E}^* + d_2\nabla\theta + d_3\boldsymbol{\varepsilon}\boldsymbol{E}^* + d_4\boldsymbol{\varepsilon}\nabla\theta + d_5\boldsymbol{\varepsilon}^2\boldsymbol{E}^* + d_6\boldsymbol{\varepsilon}^2\nabla\theta + d_7\boldsymbol{\beta}\boldsymbol{E}^* + d_8\boldsymbol{\beta}\nabla\theta + d_9\boldsymbol{\beta}^2\boldsymbol{E}^* + d_{10}\boldsymbol{\beta}^2\nabla\theta \\
& + d_{11}\boldsymbol{B}\times\boldsymbol{E}^* + d_{12}\boldsymbol{B}\times\nabla\theta + d_{13}\boldsymbol{S}\times\boldsymbol{E}^* + d_{14}\boldsymbol{S}\times\nabla\theta + d_{15}(\boldsymbol{\varepsilon}\boldsymbol{\beta} - \boldsymbol{\beta}\boldsymbol{\varepsilon})\boldsymbol{E}^* + d_{16}(\boldsymbol{\varepsilon}\boldsymbol{\beta} - \boldsymbol{\beta}\boldsymbol{\varepsilon})\nabla\theta \\
& + d_{17}[(\boldsymbol{B}\cdot\boldsymbol{B})\boldsymbol{E}^* - (\boldsymbol{B}\cdot\boldsymbol{E}^*)\boldsymbol{B}] + d_{18}[(\boldsymbol{B}\cdot\boldsymbol{B})\nabla\theta - (\boldsymbol{B}\cdot\nabla\theta)\boldsymbol{B}] + d_{19}[(\boldsymbol{S}\cdot\boldsymbol{S})\boldsymbol{E}^* - (\boldsymbol{S}\cdot\boldsymbol{E}^*)\boldsymbol{S}] \\
& + d_{20}[(\boldsymbol{S}\cdot\boldsymbol{S})\nabla\theta - (\boldsymbol{S}\cdot\nabla\theta)\boldsymbol{S}] + d_{21}[(\boldsymbol{B}\cdot\boldsymbol{E}^*)\boldsymbol{S} - (\boldsymbol{S}\cdot\boldsymbol{E}^*)\boldsymbol{B}] + d_{22}[(\boldsymbol{B}\cdot\nabla\boldsymbol{\theta})\boldsymbol{S} - (\boldsymbol{S}\cdot\nabla\theta)\boldsymbol{B}] \\
& + d_{23}[\boldsymbol{\varepsilon}(\boldsymbol{E}^*\times\boldsymbol{B}) - (\boldsymbol{\varepsilon}\boldsymbol{E}^*)\times\boldsymbol{B}] + d_{24}[\boldsymbol{\varepsilon}(\nabla\theta\times\boldsymbol{B}) - (\boldsymbol{\varepsilon}\nabla\boldsymbol{\theta})\times\boldsymbol{B}] + d_{25}[\boldsymbol{\varepsilon}(\boldsymbol{E}^*\times\boldsymbol{S}) - (\boldsymbol{\varepsilon}\boldsymbol{E}^*)\times\boldsymbol{S}] \\
& + d_{26}[\boldsymbol{\varepsilon}(\nabla\theta\times\boldsymbol{S}) - (\boldsymbol{\varepsilon}\nabla\boldsymbol{\theta})\times\boldsymbol{S}] + d_{27}[\boldsymbol{\beta}(\boldsymbol{E}^*\times\boldsymbol{B}) - (\boldsymbol{\beta}\boldsymbol{E}^*)\times\boldsymbol{B}] + d_{28}[\boldsymbol{\beta}(\nabla\theta\times\boldsymbol{B}) - (\boldsymbol{\beta}\nabla\theta)\times\boldsymbol{B}] \\
& + d_{29}[\boldsymbol{\beta}(\boldsymbol{E}^*\times\boldsymbol{S}) - (\boldsymbol{\beta}\boldsymbol{E}^*)\times\boldsymbol{S}] + d_{30}[\boldsymbol{\beta}(\nabla\theta\times\boldsymbol{S}) - (\boldsymbol{\beta}\nabla\theta)\times\boldsymbol{S}],
\end{aligned} \tag{7.35}$$

where c_i and d_i ($i = 1, 2, \ldots, 30$) are functions of θ, $\boldsymbol{X}$, and the invariants made from two 2nd order symmetric tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\beta}$, two absolute vectors $\boldsymbol{E}^*$ and $\nabla\theta$, and two axial vectors $\boldsymbol{B}$ and $\boldsymbol{S}$. The constitutive functions, c_i and d_i, are subjected to the Clausius–Duhem inequality (5.35).

From Eqs. (7.34) and (7.35), the Peltier effect (electric field producing heat flow) and the Seebeck effect (temperature gradient producing current) are clearly seen. Also, the second order vectorial effects are noticed: (1) c_3, c_5, c_7, and c_9 indicate that strains produce an anisotropic Peltier effect, (2) c_{12} shows that heat flows perpendicular to $\boldsymbol{B}$ and $\nabla\theta$, which is the Righi–Leduc effect, (3) c_{11} shows that heat flows perpendicular to $\boldsymbol{B}$ and $\boldsymbol{E}^*$, which is the Ettingshausen effect, (4) d_4, d_6, d_8, and d_{10} indicates that strains produce an anisotropic Seebeck effect, (5) d_{11} gives the Hall effect—current flows perpendicular to $\boldsymbol{B}$ and $\boldsymbol{E}^*$, and (6) d_{12} gives the Nernst effect—current flows perpendicular to $\boldsymbol{B}$ and $\nabla\theta$. Furthermore, the axial vector $\boldsymbol{S}$, which is equivalent to the anti-symmetric strain tensor representing the difference between the macro-motion and the micro-motion, has a similar effect as the magnetic flux vector $\boldsymbol{B}$. It is interesting to see that if micromorphic theory is reduced to classical continuum theory, then Eqs. (7.34) and (7.35) become

$$\begin{aligned}\boldsymbol{Q} = {} & c_1\boldsymbol{E}^* + c_2\nabla\theta + c_3\boldsymbol{\varepsilon}\boldsymbol{E}^* + c_4\boldsymbol{\varepsilon}\nabla\theta + c_5\boldsymbol{\varepsilon}^2\boldsymbol{E}^* + c_6\boldsymbol{\varepsilon}^2\nabla\theta + c_{11}\boldsymbol{B}\times\boldsymbol{E}^* + \boldsymbol{c}_{12}\boldsymbol{B}\times\nabla\theta \\ & + c_{17}[(\boldsymbol{B}\cdot\boldsymbol{B})\boldsymbol{E}^* - (\boldsymbol{B}\cdot\boldsymbol{E}^*)\boldsymbol{B}] + c_{18}[(\boldsymbol{B}\cdot\boldsymbol{B})\nabla\theta - (\boldsymbol{B}\cdot\nabla\theta)\boldsymbol{B}] + c_{23}[\boldsymbol{\varepsilon}(\boldsymbol{E}^*\times\boldsymbol{B}) - (\boldsymbol{\varepsilon}\boldsymbol{E}^*)\times\boldsymbol{B}] \\ & + c_{24}[\boldsymbol{\varepsilon}(\nabla\theta\times\boldsymbol{B}) - (\boldsymbol{\varepsilon}\nabla\boldsymbol{\theta})\times\boldsymbol{B}], \end{aligned} \tag{7.36}$$

$$\begin{aligned}\boldsymbol{J}^* = {} & d_1\boldsymbol{E}^* + d_2\nabla\theta + d_3\boldsymbol{\varepsilon}\boldsymbol{E}^* + d_4\boldsymbol{\varepsilon}\nabla\theta + d_5\boldsymbol{\varepsilon}^2\boldsymbol{E}^* + d_6\boldsymbol{\varepsilon}^2\nabla\theta + d_{11}\boldsymbol{B}\times\boldsymbol{E}^* + d_{12}\boldsymbol{B}\times\nabla\theta \\ & + d_{17}[(\boldsymbol{B}\cdot\boldsymbol{B})\boldsymbol{E}^* - (\boldsymbol{B}\cdot\boldsymbol{E}^*)\boldsymbol{B}] + d_{18}[(\boldsymbol{B}\cdot\boldsymbol{B})\nabla\theta - (\boldsymbol{B}\cdot\nabla\theta)\boldsymbol{B}] + d_{23}[\boldsymbol{\varepsilon}(\boldsymbol{E}^*\times\boldsymbol{B}) - (\boldsymbol{\varepsilon}\boldsymbol{E}^*)\times\boldsymbol{B}] \\ & + d_{24}[\boldsymbol{\varepsilon}(\nabla\theta\times\boldsymbol{B}) - (\boldsymbol{\varepsilon}\nabla\theta)\times\boldsymbol{B}]. \end{aligned} \tag{7.37}$$

The differences between Eq. (7.34) and Eq. (7.36) and between Eq. (7.35) and Eq. (7.37) are the effects due to the microstructure and the micro-motion.

Acknowledgement

The support to this work by National Science Foundation under Award Number CMS-0115868 is gratefully acknowledged.

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